

II. Spherical Coordinates

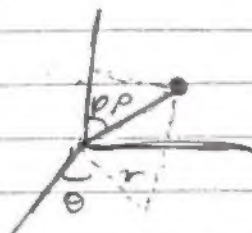
DEF: Every point in \mathbb{R}^3 lives on a sphere

We parametrize via:

ρ = distance from (x, y, z) to origin

ϕ = angle from pos x-axis to point (x, y, z)

θ = angle from pos z-axis to point (x, y, z)



$$x = r \cos \theta = \rho \sin(\phi) \cos \theta$$

$$z = \rho \cos(\phi)$$

$$y = r \sin \theta = \rho \sin(\phi) \sin(\theta)$$

Check: $\frac{d(x, y, z)}{d(\rho, \theta, \phi)} = \rho^2 \sin(\phi)$

- - - Guest Lecture - - -

Jacobian for Spherical Coordinates:
using x, y, z as above

$$J = \begin{vmatrix} \sin \phi \cos \theta & \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \sin \phi \sin \theta & \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \\ \cos \phi & -\rho \sin \phi & 0 \end{vmatrix}$$

$$= \cos \phi \begin{vmatrix} \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \end{vmatrix} + \rho \sin \phi \begin{vmatrix} \sin \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \sin \phi \sin \theta & \rho \sin \phi \cos \theta \end{vmatrix}$$

$$= \cos \phi (\rho^2 \cos \phi \sin \phi \cos^2 \theta + \rho^2 \sin \phi \cos \phi \sin^2 \theta)$$

$$+ \rho \sin \phi (\rho \sin^2 \phi \cos^2 \theta + \rho \sin^2 \phi \sin^2 \theta)$$

$$= \rho^2 \cos^2 \phi \sin \phi + \rho^2 \sin^2 \phi \sin \phi = \rho^2 \sin \phi$$

$$x^2 + y^2 + z^2 = \rho^2$$

Ex) Compute $\iiint (x^2 + y^2 + z^2)^2 dV$ where R is the solid ball of radius 5 about the origin

$$R_{\text{sph}} = \{(\rho, \phi, \theta) : 0 \leq \rho \leq 5, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi\}$$

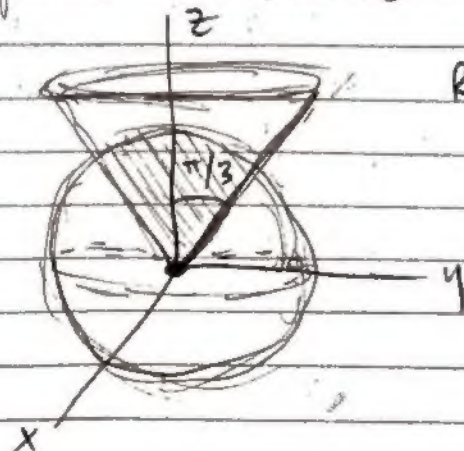
$$\int_{\rho=0}^5 \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} (\rho^2)^2 \cdot \rho^2 \sin \phi d\phi d\theta d\rho$$

Inner: $\int_{\phi=0}^{\pi} \rho^6 \sin \phi = -\rho^6 \cos \phi \Big|_0^{\pi} = 2\rho^6 - 0$

Middle: $\int_{\theta=0}^{2\pi} 2\rho^6 = 2\rho^6 \theta \Big|_{\theta=0}^{2\pi} = 4\pi \rho^6 - 0$

Outer: $\int_{\rho=0}^5 4\pi \rho^6 = 4\pi \frac{\rho^7}{7} = \boxed{4\pi \frac{5^7}{7}} - 0$

Ex: $\iiint_R (y^2 + z) dV$, R is the region above the cone w/ point at the origin and making an angle of $\pi/3$ rad w/ the positive z -axis, AND inside sphere w/ radius 2 centered at the origin



$$R: (\rho, \phi, \theta) : 0 \leq \rho \leq 2, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi/3$$

$$\int_{\rho=0}^2 \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/3} (\rho \sin \theta \sin \phi)^2 (\rho \cos(\phi)) \cdot \rho^2 \sin \theta \, d\phi \, d\theta \, d\rho$$

$$= \int_{\rho=0}^2 \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/3} \rho^5 \sin^2 \theta \sin^3 \phi \cos \phi \, d\phi \, d\theta \, d\rho$$

Inner: $\int_{\phi=0}^{\pi/3} \rho^5 \sin^2 \theta \sin^3 \phi \cos \phi \, d\phi = \rho^5 \sin^2 \theta \int_{\phi=0}^{\pi/3} \sin^3 \phi \cos \phi \, d\phi$

$$= \rho^5 \sin^2 \theta \left[\frac{\sin^4 \phi}{4} \right]_0^{\pi/3} = \rho^5 \sin^2 \theta \left(\frac{9}{64} \right)$$

Middle: $\int_{\theta=0}^{2\pi} \rho^5 \sin^2 \theta \frac{9}{64} \, d\theta = \frac{9}{64} \rho^5 \int_{\theta=0}^{2\pi} \sin^2 \theta \, d\theta$

$$= \frac{9}{128} \rho^5 \int_{\theta=0}^{2\pi} 1 - \cos(2\theta) \, d\theta = \frac{9}{128} \rho^5 \left[\theta - \frac{1}{2} \sin(2\theta) \right]_0^{2\pi}$$

$$= \frac{18\pi}{128} \rho^5$$

Outer: $\int_{\rho=0}^2 \frac{18\pi}{128} \rho^5 \, d\rho = \frac{18\pi}{128} \left[\frac{\rho^6}{6} \right]_0^2 = \frac{18\pi}{128} \left(\frac{64}{6} \right)$

$$= \frac{3\pi}{2}$$

Compute $\iiint_R 6xy \, dv$ $R = \{(x, y, z) : 0 \leq y \leq 1, y \leq x \leq 2y, 0 \leq z \leq x+y\}$

$$\int_{y=0}^1 \int_{x=y}^{2y} \int_{z=0}^{x+y} 6xy \, dz \, dx \, dy$$

inner: $\int_{z=0}^{x+y} 6xy \, dz = 6xy z \Big|_{z=0}^{x+y} = 6xy(x+y) - 0$

$$= 6x^2y + 6xy^2 = 6(x^2y + xy^2)$$

middle:

$$6 \int_{x=y}^{2y} (x^2y + xy^2) \, dx = 6 \left[\frac{x^3y}{3} + \frac{x^2y^2}{2} \right]_{x=y}^{2y}$$

$$= 6 \left[\left(\frac{8y^4}{3} + \frac{4y^4}{2} \right) - \left(\frac{y^4}{3} + \frac{y^4}{2} \right) \right]$$

$\uparrow \frac{14}{6}$ $\uparrow \frac{9}{6}$

$$= 6y^4 \left(\frac{8}{3} + \frac{4}{2} - \frac{1}{3} - \frac{1}{2} \right) = 6y^4 \left(\frac{7}{3} + \frac{3}{2} \right) = 6y^4 \frac{23}{6}$$

$$= 23y^4$$

Outer:

$$\int_{y=0}^1 23y^4 \, dy = 23 \left[\frac{y^5}{5} \right]_{y=0}^1 = \boxed{\frac{23}{5}}$$

Ex: Compute $\int_{x=0}^3 \int_{y=0}^x \int_{z=x-y}^{x+y} y \, dv$

Inner: $\int_{x-y}^{x+y} y \, dz = yz \Big|_{x-y}^{x+y} = y(x+y) - y(x-y)$
 $= yx + y^2 - xy + y^2 = 2y^2$

Middle: $\int_{y=0}^x 2y^2 \, dy = 2 \frac{y^3}{3} \Big|_{y=0}^x = \frac{2}{3} x^3$

Inner: $\int_{x=0}^3 \frac{2}{3} x^3 = \frac{2}{3} \frac{x^4}{4} \Big|_0^3 = \frac{2}{3} \frac{81}{4} = \frac{81}{6} = \boxed{\frac{27}{2}}$

Ex: $\int_R xy^2z \, dV$ where R is region bounded by $x = 4y^2 + 4z^2$ and $x = 4$

$\Rightarrow \int_{\theta=0}^{2\pi} \int_{r=0}^1 \int_{x=4r^2}^4 xy^2z \, dx \, dr \, d\theta$

$= \int_{\theta=0}^{2\pi} \int_{r=0}^1 \int_{x=4r^2}^4 x (r \cos \theta)^2 r \sin \theta \, dx \, dr \, d\theta$

Inner: $\int_{x=4r^2}^4 x r^4 \cos^2 \theta \sin \theta \, dx$

$= r^4 \cos^2 \theta \sin \theta \left[\frac{x^2}{2} \right]_{x=4r^2}^4$

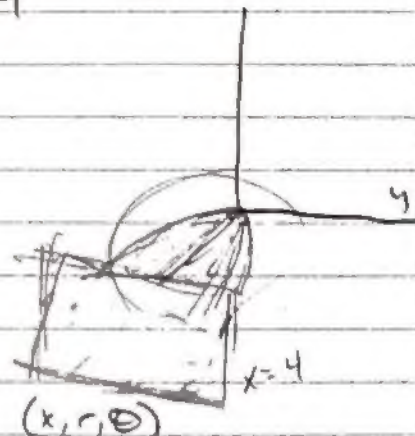
$= r^4 \cos^2 \theta \sin \theta [8 - 8r^4]$

$= 8r^4 \cos^2 \theta \sin \theta - 8r^8 \cos^2 \theta \sin \theta$

Middle:

$8 \cos^2 \theta \sin \theta \int_{r=0}^1 r^4 - r^8 \, dr = 8 \cos^2 \theta \sin \theta \left[\frac{r^5}{5} - \frac{r^9}{9} \right]_0^1$

$= \left(\frac{1}{5} - \frac{1}{9} \right) 8 \cos^2 \theta \sin \theta$



$4r^2 \leq x \leq 4$

$0 \leq r \leq 1$

$0 \leq \theta \leq 2\pi$

Outer: $(\frac{1}{8} - \frac{1}{9}) 8 \int_0^{2\pi} \sin \theta \cos^2 \theta d\theta = 0?$

Exercise: $\int_R z \, dV$ V is bounded by $x^2 + y^2 = 9$ and $y = 3x$ in the first octant

One Last Example in Spherical coordinates:

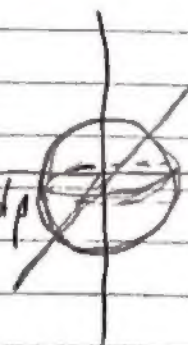
Ex: Compute the volume of the disk of radius $\alpha > 0$
 → We already did this in Cartesian coordinates but it was difficult.

Sol: In spherical coord: $D_\alpha = \{(r, \theta, \varphi) : 0 \leq r \leq \alpha, 0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \pi\}$

$$\text{Vol}(D_\alpha) = \iiint_{D_\alpha} 1 \, dV_{\text{cart}}$$

$$dV_{\text{cart}} = r^2 \sin(\varphi) \, dV_{\text{sph}}$$

$$= \iiint_{D_\alpha} 1 \cdot r^2 \sin(\varphi) \, dV_{\text{sph}} = \int_{r=0}^{\alpha} \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi} r^2 \sin(\varphi) \, d\varphi \, d\theta \, dr$$



$$= \int_0^\alpha \int_0^{2\pi} \left[-r^2 \cos(\varphi) \right]_{\varphi=0}^{\pi} d\theta \, dr = \int_0^\alpha \int_0^{2\pi} -r^2 (-1 - 1) d\theta \, dr$$

$$= \int_0^\alpha \int_0^{2\pi} 2r^2 d\theta \, dr = 2 \int_0^\alpha r^2 \left[\theta \right]_{\theta=0}^{2\pi} dr = 2 \int_0^\alpha r^2 (2\pi - 0) dr$$

$$= 4\pi \left[\frac{1}{3} r^3 \right]_{r=0}^{\alpha} = \frac{4}{3} \pi (\alpha^3 - 0)$$

$$= \frac{4}{3} \pi \alpha^3$$